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JHEP03(2009)046

(http://iopscience.iop.org/1126-6708/2009/03/046)

Published by IOP Publishing for SISSA
Received: *January 15, 2009*
Accepted: *February 15, 2009*
Published: *March 6, 2009*

A supersymmetric D_4 model for $\mu - \tau$ symmetry

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Abstract: We construct a supersymmeterized version of the model presented by Grimus and Lavoura (GL) in [1] which predicts θ_{23} maximal and $\theta_{13} = 0$ in the lepton sector. For this purpose, we extend the flavor group, which is $D_4 \times Z_2^{(aux)}$ in the original model, to $D_4 \times Z_5$. An additional difference is the absence of right-handed neutrinos. Despite these changes the model is the same as the GL model, since θ_{23} maximal and $\theta_{13} = 0$ arise through the same mismatch of D_4 subgroups, D_2 in the charged lepton and Z_2 in the neutrino sector. In our setup D_4 is solely broken by gauge singlets, the flavons. We show that their vacuum structure, which leads to the prediction of θ_{13} and θ_{23}, is a natural result of the scalar potential. We find that the neutrino mass matrix only allows for inverted hierarchy, if we assume a certain form of spontaneous CP violation. The quantity $|m_{ee}|$, measured in neutrinoless double beta decay, is nearly equal to the lightest neutrino mass m_3. The Majorana phases ϕ_1 and ϕ_2 are restricted to a certain range for $m_3 \lesssim 0.06$ eV. We discuss the next-to-leading order corrections which give rise to shifts in the vacuum expectation values of the flavons. These induce deviations from maximal atmospheric mixing and vanishing θ_{13}. It turns out that these deviations are smaller for θ_{23} than for θ_{13}.

Keywords: Neutrino Physics, Supersymmetric Standard Model, Discrete and Finite Symmetries

ArXiv ePrint: 0812.3799

doi:10.1088/1126-6708/2009/03/046

JHEP03(2009)046

Contents

1 Introduction

Experiments revealed that neutrinos have properties very distinct from the other known fermions. Their masses are much smaller and their hierarchy is much less pronounced. Unfortunately, only two mass squared differences, the solar and the atmospheric one, are known from experiments [2]

$$\Delta m_{21}^2 = \left(7.65^{+0.46}_{-0.40}\right) \cdot 10^{-5} \text{ eV}^2 \quad \text{and} \quad |\Delta m_{31}^2| = \left(2.40^{+0.24}_{-0.22}\right) \cdot 10^{-3} \text{ eV}^2 \qquad (2\,\sigma) \qquad (1.1)$$

where Δm_{ij}^2 denotes $m_i^2 - m_j^2$ with $m_{i,j}$ being the neutrino masses. As the sign of Δm_{31}^2 is unknown it is not clear whether neutrinos follow a normal or an inverted hierarchy. Even more surprising than the neutrino masses is the fact that leptons have a very peculiar mixing pattern with two large mixing angles and one small one. The experimental measurements [2]

$$\begin{array}{lll} \sin^2\theta_{13} \leq 0.040\ , & \theta_{13} \leq 11.5^\circ\ , & \theta_{13} \leq 0.2\ , \\ \sin^2\theta_{23} = 0.50^{+0.13}_{-0.11}\ , & \theta_{23} = (45.0^{+7.5}_{-6.4})^\circ\ , & \theta_{23} = 0.785^{+0.13}_{-0.11}\ , \\ \sin^2\theta_{12} = 0.304^{+0.046}_{-0.034}\ , & \theta_{12} = (33.5^{+2.8}_{-2.2})^\circ\ , & \theta_{12} = 0.58^{+0.05}_{-0.04}\ , \end{array} \qquad (2\,\sigma) \qquad (1.2)$$

allow for the possibility that the atmospheric mixing angle θ_{23} is maximal and the reactor mixing angle θ_{13} vanishes. These values can be deduced from a neutrino mass matrix which is $\mu-\tau$ symmetric [3], i.e. does not change its form if second and third columns and rows are interchanged, in the charged lepton mass basis. At the same time the solar mixing angle θ_{12} is left undetermined. An even more constraining pattern is the one of tri-bimaximal (TB) mixing [4] in which, apart from $\theta_{23}=\pi/4$ and $\theta_{13}=0$, also θ_{12} is fixed by $\sin^2\theta_{12}=1/3$. Both patterns have been subject to extensive studies in order to find a theoretical explanation. The most promising one seems to be the assumption of an additional flavor symmetry which is responsible for such a mixing pattern. For TB mixing the simplest models are based on the tetrahedral group A_4 [5],[1] while for $\mu-\tau$ symmetry even smaller groups are appropriate such as the dihedral groups $D_3 \cong S_3$ [7] and D_4 [1].

An interesting observation which has been made first in the A_4 models and then also in the models predicting $\mu-\tau$ symmetry with a dihedral flavor group is the fact that the VEVs of a certain subset of scalar fields $\{\phi_e\}$, coupling only to charged leptons at leading order (LO), preserve one subgroup of the original symmetry, while another set of scalars $\{\phi_\nu\}$, coupling only to neutrinos at LO, breaks the flavor group to a different subgroup. This mismatch can be regarded as an intuitive explanation for sizable mixings in the lepton sector.[2,3] For more general considerations on the origin of a certain mixing pattern see [8].

One of the main issues in these models is the vacuum alignment of the flavor symmetry breaking fields, since without a special alignment the mixing pattern is merely a result of parameter tuning. In a class of A_4 models [12] this alignment is achieved in a supersymmetric framework where the scalars transforming non-trivially under flavor are only gauge singlets. Their vacuum expectation values (VEVs) are driven by another set of gauge singlets, the driving fields. This mechanism to align the flavon VEVs is also used in models with other discrete [13] and with continuous flavor symmetries [14].

The D_4 model [1] by Grimus and Lavoura (GL), which successfully predicts $\mu-\tau$ symmetry in a natural way including a profound explanation for the vacuum alignment, is non-supersymmetric in its original version. For several reasons, it would be desirable to supersymmeterize this model. In doing this, it is of advantage to break the flavor and the gauge group (spontaneously) by separate sets of scalar fields, flavons and Higgs doublets. In this paper we present such a supersymmeterized version in which the vacuum alignment is achieved in a similar way as in [12]. We arrive at $\theta_{23}=\pi/4$ and $\theta_{13}=0$ at LO and analyze the next-to-leading order (NLO) effects which will perturb this result in a particular way. In the minimal supersymmetric extension presented here the flavor symmetry D_4 is accompanied by a Z_5 symmetry which plays a similar role as $Z_2^{(aux)}$ in the original model. Furthermore, our model does not incorporate right-handed neutrinos so that the light neutrino masses stem from the dimension-5 operator $lh_u lh_u/\Lambda$. Despite these changes the model is essentially a supersymmeterized version of the GL model, since the prediction of maximal θ_{23} and vanishing θ_{13} is still due to the fact that we preserve, at

[1] From a group theoretical point of view S_4, the permutation group of four distinct objects, might be even more appropriate [6].

[2] A similar observation has been made concerning the Cabibbo angle in the quark sector whose value might be the result of a non-trivial breaking of a dihedral group such as D_7 or D_{14} [8–10].

[3] Notice that there are also models which can predict lepton mixings without preserving non-trivial subgroups in all sectors of the theory, for instance [11].

LO, a D_2 subgroup of D_4 in the charged lepton and a Z_2 subgroup in the neutrino sector. Since this Z_2 group is not contained in the D_2 group of the charged lepton sector, D_4 is completely broken in the whole theory. Apart from predicting the value of θ_{13} and θ_{23}, the original GL model also predicts that neutrinos are normally ordered and that the effective Majorana mass of neutrinoless $\beta\beta$ ($0\nu\beta\beta$) decay is equal to $|m_{ee}| = m_1 m_2/m_3$. These predictions result from the fact that in the GL model not all fields are present which are allowed to have a non-vanishing VEV in accordance with a preserved Z_2 subgroup in the neutrino sector. Here we will include all flavons allowed by the symmetry principle so that we still predict $\theta_{23} = \pi/4$ and $\theta_{13} = 0$, but now can accommodate both mass hierarchies. In this respect our results are analogous to another model by GL predicting θ_{13} and θ_{23} with the help of the dihedral group $D_3 \cong S_3$ [7].

In our phenomenological study we concentrate on the possibility of a certain type of spontaneous CP violation in order to make the model more predictive. We find that neutrinos then have to have an inverted hierarchy and $|m_{ee}| \approx m_3$ holds. Furthermore, the Majorana phases $\phi_{1,2}$ can only take values in a limited range for $m_3 \lesssim 0.06$ eV. If we additionally remove one of the flavons from the model (this is analogous to what is done in [1]), the three parameters of the model in the neutrino sector are determined by the three measured quantities, Δm^2_{21}, $|\Delta m^2_{31}|$ and θ_{12}, and the Majorana phases are predicted to be $\phi_1 = \pi/2$ and $\phi_2 = 0$. The NLO corrections coming from the inclusion of operators with one more flavon lead to deviations from $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. It turns out that $\theta_{23} - \pi/4$ is much smaller than θ_{13}.

In [15] it has already been attempted to build a D_4 model in which the Higgs doublets transforming non-trivially under the flavor group are replaced by flavons. Since this model is non-supersymmetric the vacuum alignment problem is not straightforward to solve and indeed one has to require that one of the quartic couplings in the potential vanishes. However, such an assumption will not be stable against corrections and has to be considered as a severe tuning. In a second version of this model [16] which is supersymmetric, the potential is not studied such that the question of the vacuum alignment also remains open. Hence, a successful supersymmeterization of the original D_4 model by GL still does not exist.

The paper is organized as follows: in section 2 we repeat the necessary group theory of D_4 and the properties of the subgroups relevant in the D_4 model. Section 3 contains the LO results for the lepton masses and mixings as well as the flavon potential. In the following section the NLO corrections are studied. We summarize our results and give a short outlook in section 5. In the two appendices we treat additional group theoretical aspects of the model.

2 Group theory of D_4

In this section we briefly review basic features of the dihedral group D_4. Its order is eight, and it has five irreducible representations which we denote as $\underline{\mathbf{1}}_{\mathbf{i}}$, i $= 1, \ldots, 4$ and $\underline{\mathbf{2}}$. All of them are real and only $\underline{\mathbf{2}}$ is faithful. The group is generated by the two generators A and

	classes				
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
G	$\mathbb{1}$	A	A^2	B	A B
$^\circ\mathcal{C}_i$	1	2	1	2	2
$^\circ\mathrm{h}_{\mathcal{C}_i}$	1	4	2	2	2
$\underline{\mathbf{1_1}}$	1	1	1	1	1
$\underline{\mathbf{1_2}}$	1	1	1	-1	-1
$\underline{\mathbf{1_3}}$	1	-1	1	1	-1
$\underline{\mathbf{1_4}}$	1	-1	1	-1	1
$\underline{\mathbf{2}}$	2	0	-2	0	0

Table 1. Character table of the group D_4. $\mathcal{C}_i$ are the classes of the group, $^\circ\mathcal{C}_i$ is the order of the i^{th} class, i.e. the number of distinct elements contained in this class, $^\circ\mathrm{h}_{\mathcal{C}_i}$ is the order of the elements S in the class $\mathcal{C}_i$, i.e. the smallest integer (> 0) for which the equation $S^{\,^\circ\mathrm{h}_{\mathcal{C}_i}} = \mathbb{1}$ holds. Furthermore the table contains one representative for each class $\mathcal{C}_i$ given as product of the generators A and B of the group.

B which can be chosen as [17]

$$\mathrm{A} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1}$$

for $\underline{\mathbf{2}}$. Note that A is a complex matrix, although $\underline{\mathbf{2}}$ is a real representation. For $(a_1, a_2)^T \sim \underline{\mathbf{2}}$ therefore $(a_2^\star, a_1^\star)^T$ transforms as $\underline{\mathbf{2}}$ under D_4. The generators of the one-dimensional representations can be found in the character table, displayed in table 1. The generators fulfill the relations

$$\mathrm{A}^4 = \mathbb{1} \quad , \quad \mathrm{B}^2 = \mathbb{1} \quad \text{and} \quad \mathrm{ABA} = \mathrm{B} \, . \tag{2.2}$$

The product rules for $\underline{\mathbf{1_i}}$ are the following

$$\begin{array}{lll} \underline{\mathbf{1_i}} \times \underline{\mathbf{1_i}} = \underline{\mathbf{1_1}} \, , & \underline{\mathbf{1_1}} \times \underline{\mathbf{1_i}} = \underline{\mathbf{1_i}} & \text{for } \mathrm{i} = 1, \dots, 4 \, , \\ \underline{\mathbf{1_2}} \times \underline{\mathbf{1_3}} = \underline{\mathbf{1_4}} \, , & \underline{\mathbf{1_2}} \times \underline{\mathbf{1_4}} = \underline{\mathbf{1_3}} & \text{and } \underline{\mathbf{1_3}} \times \underline{\mathbf{1_4}} = \underline{\mathbf{1_2}} \, . \end{array} \tag{2.3}$$

For $s_i \sim \underline{\mathbf{1_i}}$ and $(a_1, a_2)^T \sim \underline{\mathbf{2}}$ we find

$$\begin{pmatrix} s_1 a_1 \\ s_1 a_2 \end{pmatrix} \sim \underline{\mathbf{2}} \, , \quad \begin{pmatrix} s_2 a_1 \\ -s_2 a_2 \end{pmatrix} \sim \underline{\mathbf{2}} \, , \quad \begin{pmatrix} s_3 a_2 \\ s_3 a_1 \end{pmatrix} \sim \underline{\mathbf{2}} \quad \text{and} \quad \begin{pmatrix} s_4 a_2 \\ -s_4 a_1 \end{pmatrix} \sim \underline{\mathbf{2}} \, .$$

The product $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$ decomposes into the four singlets which read for $(a_1, a_2)^T$, $(b_1, b_2)^T \sim \underline{\mathbf{2}}$

$$a_1 b_2 + a_2 b_1 \sim \underline{\mathbf{1_1}} \quad , \quad a_1 b_2 - a_2 b_1 \sim \underline{\mathbf{1_2}} \quad , \quad a_1 b_1 + a_2 b_2 \sim \underline{\mathbf{1_3}} \quad \text{and} \quad a_1 b_1 - a_2 b_2 \sim \underline{\mathbf{1_4}} \, .$$

More general formulae for generators, Kronecker products and Clebsch Gordan coefficients can be found, for example, in [9, 18]. Notice that our group basis does not coincide with the one chosen by GL in [1]. Therefore, the mass matrices shown below have another appearance, especially the charged lepton mass matrix is not diagonal in our basis. However, the

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prediction of the mixing angles does not depend on the chosen group basis. In appendix A we explicitly discuss the correlation between our basis and the one found in [1].

All subgroups of D_4 are abelian: $Z_2 \cong D_1$, Z_4 and $D_2 \cong Z_2 \times Z_2$. We are interested here in Z_2 subgroups which are generated by $\mathrm{B\,A}^m$ with $m = 0, \ldots, 3$ and the D_2 subgroup generated by A^2 and BA. In order to see that $\mathrm{B\,A}^m$ gives a Z_2 group note that

$$(\mathrm{B\,A}^m)^2 = \mathrm{B\,A}^m \mathrm{B\,A}^m = \mathrm{B\,A}^{m-1} \mathrm{B\,A}^{m-1} = \cdots = \mathrm{B}^2 = \mathbb{1}$$

holds, if eq. (2.2) is used. Similarly, one finds for A^2 and BA

$$(\mathrm{A}^2)^2 = \mathrm{A}^4 = \mathbb{1} \quad \text{and} \quad (\mathrm{BA})^2 = \mathrm{BABA} = \mathrm{B}^2 = \mathbb{1}$$

by using again the generator relations. Obviously, A^2 and BA are not equal (in general) and thus they generate different Z_2 subgroups. Additionally, we have to check that A^2 and BA commute

$$\mathrm{A}^2\mathrm{BA} = \mathrm{A}^3\mathrm{BA}^2 = \mathrm{A}^4\mathrm{BA}^3 = \mathrm{BAA}^2 \ .$$

All this shows that A^2 and BA generate a $Z_2 \times Z_2$ group which is isomorphic to a D_2 group. The other non-trivial element of the D_2 group is BA^3. Thus, one could also use the two elements A^2 and BA^3 to generate this D_2 group. However, we follow the convention to use as generators A^2 and the element BA^p with p being the smallest possible natural number. The Z_2 symmetry given through BA^m is left unbroken by a non-vanishing VEV of a singlet transforming as $\underline{\mathbf{1}}_{\mathbf{3}}$ if m is even and of one transforming as $\underline{\mathbf{1}}_{\mathbf{4}}$ for m being odd. Additionally, it is left intact by fields $\psi_{1,2}$ forming a doublet, if their VEVs have the following structure

$$\begin{pmatrix} \langle\psi_1\rangle \\ \langle\psi_2\rangle \end{pmatrix} \propto \begin{pmatrix} \mathrm{e}^{-\frac{\pi i m}{2}} \\ 1 \end{pmatrix} \ . \tag{2.4}$$

For preserving the D_2 group generated by A^2 and BA only singlets in $\underline{\mathbf{1}}_{\mathbf{4}}$ are allowed to have a non-vanishing VEV. Especially, no fields forming a doublet under D_4 should acquire a VEV. Clearly, in all cases singlets in the trivial representation of D_4, $\underline{\mathbf{1}}_{\mathbf{1}}$, are allowed to have a non-vanishing VEV. Note also that in none of the cases a field transforming as $\underline{\mathbf{1}}_{\mathbf{2}}$ can acquire a non-zero VEV. Since we concentrate on the D_2 subgroup induced by A^2 and BA, the Z_2 subgroup has to be generated by BA^m with m being even in order not to be a subgroup of the D_2 group. Only then the mismatch between the two subgroups is achieved. The choice of m, $m = 0$ or $m = 2$, depends on the relative sign between $\langle\psi_1\rangle$ and $\langle\psi_2\rangle$ for two fields $\psi_{1,2} \sim \underline{\mathbf{2}}$.

3 The model at leading order

We augment the Minimal Supersymmetric Standard Model (MSSM) by the flavor symmetry $D_4 \times Z_5$. As mentioned above, the non-trivial breaking of D_4 is responsible for maximal atmospheric mixing and vanishing θ_{13}, while Z_5 is necessary to separate the charged lepton and the neutrino sector. The model contains three left-handed lepton doublets l_i, the three right-handed charged leptons e^c_i, the MSSM Higgs doublets $h_{u,d}$ and two sets of flavons $\{\chi_e, \varphi_e\}$ and $\{\chi_\nu, \varphi_\nu, \psi_{1,2}\}$ which break D_4 in the charged lepton and the neutrino sector, respectively. The transformation properties of these fields are collected in table 2.

Field	l_1	$l_{2,3}$	e^c_1	$e^c_{2,3}$	h_u	h_d	χ_e	φ_e	χ_ν	φ_ν	$\psi_{1,2}$
D_4	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{1}}_\mathbf{4}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{1}}_\mathbf{3}$	$\underline{\mathbf{2}}$
Z_5	ω	ω	1	1	ω^3	ω	ω^3	ω^3	ω^2	ω^2	ω^2

Table 2. Particle content of the model. l_i denotes the three left-handed lepton $\mathrm{SU}(2)_L$ doublets, e^c_i are the right-handed charged leptons and $h_{u,d}$ are the MSSM Higgs doublets. The flavons χ_e, φ_e, χ_ν, φ_ν and $\psi_{1,2}$ only transform under $D_4 \times Z_5$. The phase factor ω is $e^{\frac{2\pi i}{5}}$.

3.1 Fermion masses

The invariance of the charged lepton and neutrino mass terms under the flavor group $D_4 \times Z_5$ requires the presence of at least one flavon. Thus, charged lepton masses are generated by non-renormalizable operators only. In a model which treats quarks as well this allows the explanation of the small τ mass compared to the top quark mass without relying on a large value of $\tan\beta = \langle h_u \rangle / \langle h_d \rangle = v_u/v_d$. The neutrinos receive Majorana masses through the dimension-5 operator $l h_u l h_u / \Lambda$ which can be made invariant under the flavor group by coupling to a flavon. The part of the superpotential giving lepton masses reads at LO

$$\begin{aligned} w_l = & \, y_1^e \chi_e l_1 e_1^c \frac{h_d}{\Lambda} + y_2^e \chi_e (l_2 e_3^c + l_3 e_2^c) \frac{h_d}{\Lambda} + y_3^e \varphi_e (l_2 e_2^c - l_3 e_3^c) \frac{h_d}{\Lambda} \\ & + y_1 \chi_\nu l_1 l_1 \frac{h_u^2}{\Lambda^2} + y_2 l_1 (l_2 \psi_2 + l_3 \psi_1) \frac{h_u^2}{\Lambda^2} + y_2 (l_2 \psi_2 + l_3 \psi_1) l_1 \frac{h_u^2}{\Lambda^2} + y_3 \varphi_\nu (l_2 l_2 + l_3 l_3) \frac{h_u^2}{\Lambda^2} \\ & + y_4 \chi_\nu (l_2 l_3 + l_3 l_2) \frac{h_u^2}{\Lambda^2} . \end{aligned} \tag{3.1}$$

Λ is the cutoff scale of the theory whose order of magnitude is determined by the scale of the light neutrino masses, see below. For the moment we assume that the flavons χ_e and φ_e acquire the VEVs

$$\langle \varphi_e \rangle = u_e \quad \text{and} \quad \langle \chi_e \rangle = w_e \, . \tag{3.2}$$

As discussed in section 2 these VEVs break D_4 down to D_2 generated by A^2 and BA in the charged lepton sector. The VEVs of the flavons coupling only to neutrinos at LO, are of the form

$$\langle \varphi_\nu \rangle = u \; , \quad \langle \chi_\nu \rangle = w \; , \quad \begin{pmatrix} \langle \psi_1 \rangle \\ \langle \psi_2 \rangle \end{pmatrix} = v \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \tag{3.3}$$

and therefore leave a Z_2 subgroup, generated by B, unbroken. As mentioned, the equality of the VEVs of $\langle \psi_1 \rangle$ and $\langle \psi_2 \rangle$ is crucial. As will be discussed in section 3.3, the vacuum structure in eq. (3.2) and eq. (3.3) is a natural result of the minimization of the flavon potential. We obtain the following fermion mass matrices, when inserting the flavon VEVs and $\langle h_{u,d} \rangle = v_{u,d}$

$$M_l = \frac{v_d}{\Lambda} \begin{pmatrix} y_1^e w_e & 0 & 0 \\ 0 & y_3^e u_e & y_2^e w_e \\ 0 & y_2^e w_e & -y_3^e u_e \end{pmatrix} \quad \text{and} \quad M_\nu = \frac{v_u^2}{\Lambda^2} \begin{pmatrix} y_1 w & y_2 v & y_2 v \\ y_2 v & y_3 u & y_4 w \\ y_2 v & y_4 w & y_3 u \end{pmatrix} . \tag{3.4}$$

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Thereby, the left-handed fields are on the left-hand and the right-handed fields on the right-hand side for M_l. The matrix $M_l\, M_l^\dagger$ is diagonalized through the unitary matrix U_l, i.e. $U_l^\dagger\, M_l\, M_l^\dagger\, U_l$ is diagonal. U_l acts on the left-handed charged lepton fields and is given by

$$U_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/4}/\sqrt{2} & e^{-i\pi/4}/\sqrt{2} \\ 0 & e^{-i\pi/4}/\sqrt{2} & e^{i\pi/4}/\sqrt{2} \end{pmatrix} . \tag{3.5}$$

For the masses of the charged leptons we find

$$m_e = \frac{v_d}{\Lambda} |y_1^e w_e| \, , m_\mu = \frac{v_d}{\Lambda} |y_3^e u_e + i y_2^e w_e| \quad \text{and} \quad m_\tau = \frac{v_d}{\Lambda} |y_3^e u_e - i y_2^e w_e| \, . \tag{3.6}$$

In order to arrive at non-degenerate masses for the μ and the τ lepton either $y_3^e u_e$ or $y_2^e w_e$ has to be non-real indicating CP violation in the Yukawa couplings and/or flavon VEVs. For m_τ being around 2 GeV we find that for small $\tan\beta$ - corresponding to v_d of the order of 100 GeV - the ratio of the flavon VEVs u_e and w_e over the cutoff scale Λ should fulfill [4]

$$\frac{u_e}{\Lambda} , \frac{w_e}{\Lambda} \sim \lambda^2 \approx 0.04 \tag{3.7}$$

with λ being the Cabibbo angle. The smallness of the ratio m_e/m_τ is in this model only explained by the assumption of a small enough coupling y_1^e. Similarly, m_μ/m_τ enforces a certain cancellation between the two contributions $y_3^e u_e$ and $i y_2^e w_e$ in m_μ. In [1] these problems have been solved by the assumption that the electron couples to a Higgs field different from those coupling to the μ and the τ lepton and by an additional symmetry which leads to $m_\mu = 0$, if it is unbroken, respectively.

The neutrino mass matrix in the charged lepton mass basis reads (indicated by a prime ($'$))

$$M_\nu' = U_l^\dagger M_\nu U_l^* = \frac{v_u^2}{\Lambda^2} \begin{pmatrix} y_1 w & y_2 v & y_2 v \\ y_2 v & y_4 w & y_3 u \\ y_2 v & y_3 u & y_4 w \end{pmatrix} . \tag{3.8}$$

As M_ν' is $\mu-\tau$ symmetric, it immediately follows that the lepton mixing angle θ_{13} vanishes and θ_{23} is maximal. The solar mixing angle θ_{12} is not predicted, but in general expected to be large. Also the Majorana phases $\phi_{1,2}$ are not constrained. The lepton mixing matrix is of the form

$$U_{\text{MNS}} = \text{diag}(e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3}) \cdot \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\frac{\sin\theta_{12}}{\sqrt{2}} & \frac{\cos\theta_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sin\theta_{12}}{\sqrt{2}} & \frac{\cos\theta_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \text{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}) \, . \tag{3.9}$$

The Majorana phases $\phi_{1,2} = \alpha_{1,2}/2$ can be extracted from U_{MNS} by bringing it into the standard form [19]. Due to the additional factor 1/2 the phases $\phi_{1,2}$ vary between 0 and π. Assuming that all flavon VEVs are of the same size, the estimate in eq. (3.7) also holds for the VEVs of the flavons χ_ν, φ_ν and $\psi_{1,2}$. For small $\tan\beta$, i.e. $v_u \approx v_d \approx 100$ GeV, a

[4] Although not excluded, there is no obvious reason to assume that there is a large hierarchy among the different flavon VEVs. In general, these are correlated through the parameters of the flavon potential.

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light neutrino mass scale between $\sqrt{|\Delta m_{31}^2|} \approx 0.05\,\mathrm{eV}$ and $1\,\mathrm{eV}$ fixes the range of the cutoff scale Λ to be

$$4 \cdot 10^{11}\ \mathrm{GeV} \lesssim \Lambda \lesssim 8 \cdot 10^{12}\ \mathrm{GeV}\,. \tag{3.10}$$

As shown in section 3.3, we can assume that CP is only spontaneously violated in this model by imaginary VEVs w_e and w of χ_e and χ_ν. Thus, apart from w_e and w all other parameters, i.e. couplings and VEVs, are real in the following. According to eq. (3.6) an imaginary w_e allows the μ and the τ lepton mass to be non-degenerate. In the neutrino sector only the VEV w of χ_ν is imaginary, whereas all other entries in M'_ν are real, so that the matrix in eq. (3.8) can be written as

$$M'_\nu = \frac{v_u^2}{\Lambda}\frac{v}{\Lambda}\begin{pmatrix} i\,s & t & t \\ t & i\,x & z \\ t & z & i\,x \end{pmatrix} \tag{3.11}$$

where we define the real parameters

$$s = y_1 \frac{\mathrm{Im}(w)}{v}\,, \quad t = y_2\,, \quad x = y_4 \frac{\mathrm{Im}(w)}{v} \quad \text{and} \quad z = y_3 \frac{u}{v}\,. \tag{3.12}$$

3.2 Phenomenology

In the following we analyze the phenomenology of this model. For the eigenvalues of $M'_\nu M'^\dagger_\nu$ we find

$$\begin{aligned} m_{2,1}^2 &= \frac{1}{2}\left(\frac{v_u^2}{\Lambda}\right)^2\left(\frac{v}{\Lambda}\right)^2\Big[s^2+4t^2+x^2+z^2 \\ &\qquad \pm\sqrt{(s-x)^2(8t^2+(s+x)^2)+2(4t^2+x^2-s^2)z^2+z^4}\,\Big] \\ \text{and}\quad m_3^2 &= \left(\frac{v_u^2}{\Lambda}\right)^2\left(\frac{v}{\Lambda}\right)^2\left(x^2+z^2\right)\,. \end{aligned} \tag{3.13}$$

This assignment of the eigenvalues is unambiguous, since $m_2^2 > m_1^2$ is experimentally known and the eigenvalue corresponding to the eigenvector $(0,1,-1)^T$ can only be m_3^2. The solar mixing angle θ_{12} is found to depend on s, t, x and z in the following way

$$\tan 2\theta_{12} = \frac{2\sqrt{2}\,|t|\,\sqrt{(s-x)^2+z^2}}{x^2+z^2-s^2}\,. \tag{3.14}$$

Before discussing the general case with unconstrained parameters s, t, x and z we comment on the special case in which z vanishes, since then the model contains three real parameters which can be determined by the three experimental quantities Δm_{21}^2, $|\Delta m_{31}^2|$ and θ_{12}. According to eq. (3.12) either y_3 or u have to vanish for $z=0$ to hold. Assuming that y_3 is zero however has to be regarded as fine-tuning. In contrast to that, a vanishing VEV u can be explained either through the absence of the flavon φ_ν from the model or through a flavon potential which only allows configurations with $u=0$ to be minima. The neutrino mass m_3 is then proportional to $|x|$. From eq. (3.13) and eq. (3.14) we can derive for $z=0$

$$m_3^2 = -\frac{1}{4}\frac{\cos^4\theta_{12}}{\sin^2\theta_{12}}\frac{(\Delta m_{21}^2+\Delta m_{31}^2(\tan^4\theta_{12}-1))^2}{\Delta m_{31}^2(1+\tan^2\theta_{12})-\Delta m_{21}^2}\,. \tag{3.15}$$

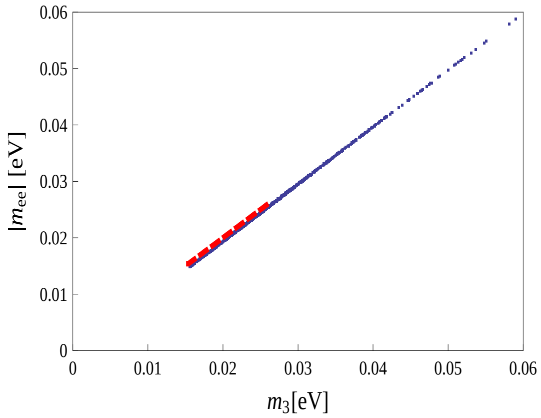

Figure 1. $|m_{ee}|$ plotted against m_3 for $z \neq 0$. The dashed (red) line indicates the results for $z = 0$. Mass squared differences and the solar mixing angle are in the allowed 2σ ranges [2]. As one can see, $|m_{ee}|$ and m_3 have nearly the same value. Additionally, one finds that m_3 has a lower bound around 0.015 eV. For $z = 0$ we also find an upper bound on m_3.

Neglecting the solar mass squared difference we can simplify this expression to

$$m_3^2 \approx -\Delta m_{31}^2 \cot^2 2\theta_{12} \ . \tag{3.16}$$

Eq. (3.16) shows that $\Delta m_{31}^2 < 0$, i.e. the neutrinos have to have an inverted hierarchy. Note that similar results can also be found in [20]. A relation analogous to eq. (3.15) can be found for $|m_{ee}|$ measured in $0\nu\beta\beta$ decay experiments. Note that $|m_{ee}|$ is proportional to $|s|$ due to eq. (3.11) and can be written in terms of m_3, $\tan\theta_{12}$ and the mass squared differences as

$$|m_{ee}|^2 = m_3^2 \frac{(\Delta m_{21}^2(1-2\tan^2\theta_{12}) + \Delta m_{31}^2(\tan^4\theta_{12}-1))^2}{(\Delta m_{21}^2 + \Delta m_{31}^2(\tan^4\theta_{12}-1))^2} \ . \tag{3.17}$$

In the limit of vanishing solar mass splitting we find

$$|m_{ee}| \approx m_3 \ . \tag{3.18}$$

Taking the best-fit values $\Delta m_{21}^2 = 7.65 \cdot 10^{-5}\,\text{eV}^2$, $\Delta m_{31}^2 = -2.40 \cdot 10^{-3}\,\text{eV}^2$ and $\sin^2\theta_{12} = 0.304$ [2] we obtain $s \approx 0.02075$, $t \approx 0.03502$, $x \approx 0.02146$ [5] for $v_u \approx 100$ GeV, $\Lambda \approx 4 \cdot 10^{11}$ GeV and $v/\Lambda \approx \lambda^2 \approx 0.04$. The neutrino masses are $m_1 \approx 0.05348$ eV, $m_2 \approx 0.05419$ eV and $m_3 \approx 0.02146$ eV. Their sum $\sum m_i \approx 0.1291$ eV lies below the upper bound required from cosmological data [21]. $|m_{ee}|$ equals 0.02075 eV which might be detectable in the future [22]. The two Majorana phases $\phi_{1,2}$ are $\phi_1 = \pi/2$ and $\phi_2 = 0$. For tritium β decay we find $m_\beta \approx 0.05370$ eV which is about a factor of six smaller than the expected sensitivity of the KATRIN experiment [23].

Turning to the general case with $z \neq 0$ we first observe that also in this case the light neutrinos have to have an inverted hierarchy. To see this let us assume that the matrix

[5] Actually we find four solutions which all lead to the same absolute values, but to different signs for s, t and x, with the constraint that s and x have the same sign.

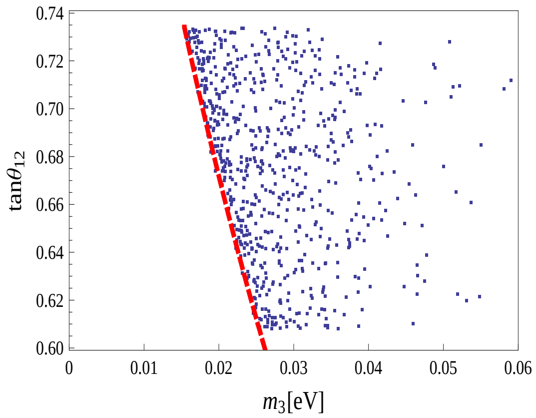

Figure 2. $\tan\theta_{12}$ plotted against m_3 for non-vanishing z. Again the dashed (red) line indicates $z = 0$ (assuming the best-fit value for the atmospheric mass squared difference) and gives a lower bound for $z \neq 0$. Apart from that the results for $\tan\theta_{12}$ are only constrained by the requirement that they are within the experimental 2σ ranges [2], $0.61 \lesssim \tan\theta_{12} \lesssim 0.73$.

in eq. (3.11) would allow the neutrinos to be normally ordered, i.e. $m_3 > m_1$ as well as $m_3 > m_2$. From $m_3^2 - m_2^2 > 0$ then follows

$$x^2 + z^2 - s^2 - 4t^2 - \sqrt{(s-x)^2(8t^2 + (s+x)^2) + 2(4t^2 + x^2 - s^2)z^2 + z^4} > 0\,. \tag{3.19}$$

From this we can deduce

$$x^2 + z^2 > s^2 + 4t^2 \quad \text{and} \quad 16t^2(t^2 + x(s-x) - z^2) > 0\,. \tag{3.20}$$

Rearranging the first inequality and taking $t \neq 0$ (otherwise θ_{12} is zero) for the second one, we get

$$x^2 - s^2 > 4t^2 - z^2 \quad \text{and} \quad t^2 - z^2 > x(x-s)\,. \tag{3.21}$$

The sum of these inequalities leads to

$$s(x-s) > 3t^2 > 0\,. \tag{3.22}$$

From eq. (3.22) we see that s and x have the same sign, while $x^2 > s^2$, hence $x(x-s) > s(x-s)$. Combining eq. (3.21) and eq. (3.22), we find $t^2 - z^2 > 3t^2$, an obvious contradiction. Thus, the neutrinos cannot be normally ordered as assumed by $m_3^2 > m_2^2$. Instead we always have $m_2^2 > m_3^2$ which is only possible in case of an inverted hierarchy. Note that it is a priori not clear that also m_1 is larger than m_3, since the size of the mass squared differences has to be tuned so that $\Delta m_{21}^2 \ll |\Delta m_{31}^2|$. In fact, Δm_{21}^2 is given by

$$\Delta m_{21}^2 = \left(\frac{v_u^2}{\Lambda}\right)^2 \left(\frac{v}{\Lambda}\right)^2 \sqrt{(s-x)^2(8t^2 + (s+x)^2) + 2(4t^2 + x^2 - s^2)z^2 + z^4}\,. \tag{3.23}$$

It vanishes, if $z = 0$ and $s = x$. Thus, $\Delta m_{21}^2 \ll |\Delta m_{31}^2|$ holds, if these equalities are nearly met. As noted, the vanishing of z can be made a natural result of the model. The near

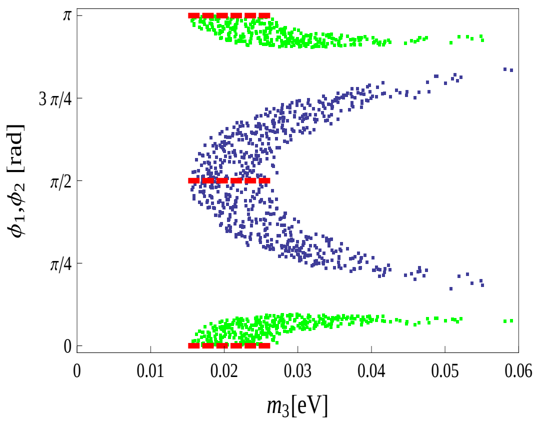

Figure 3. The Majorana phases ϕ_1 (blue/darker gray) and ϕ_2 (green/lighter gray) plotted against the lightest neutrino mass m_3 for non-vanishing z. The values for $z = 0$, $\phi_1 = \frac{\pi}{2}$, $\phi_2 = 0$, are displayed by dashed (red) lines. Notice that the results for $z \neq 0$ are centered around these values. The measured quantities, Δm^2_{21}, $|\Delta m^2_{31}|$ and θ_{12}, are within the 2σ ranges [2].

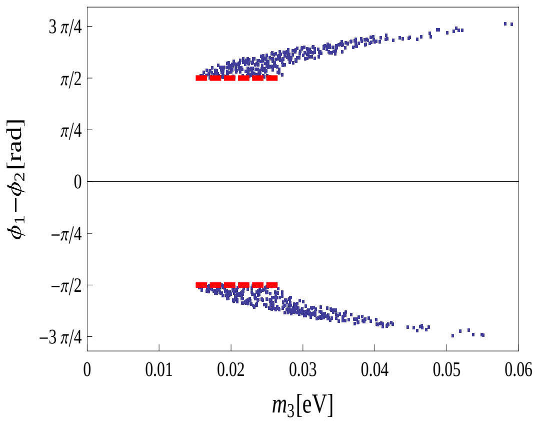

Figure 4. Phase difference $\phi_1 - \phi_2$ against m_3 for $z \neq 0$. The case $z = 0$, $|\phi_1 - \phi_2| = \pi/2$, is given by the dashed (red) lines. As one can see, $|\phi_1 - \phi_2|$ is restricted to the interval $[\pi/2, 3\pi/4]$ for $m_3 \lesssim 0.06\,\mathrm{eV}$. Its deviation from $\pi/2$ increases with increasing m_3. Again, the mass squared differences and θ_{12} are within the experimentally allowed 2σ ranges [2].

equality $s \approx x$ however has to be regarded as a certain tuning of the couplings y_1 and y_4, see eq. (3.12).

We study the general case $z \neq 0$ with a numerical analysis. To fix the light neutrino mass scale we adjust the resulting solar mass squared difference to its best-fit value. At the same time the atmospheric mass squared difference and the mixing angle θ_{12} have to be within the allowed 2σ ranges [2]. First, we note that our numerical results confirm that z has to be in general smaller than the parameters s, t and x and that s and x have to have nearly the same value. In figure 1 we plotted $|m_{ee}|$ against the lightest neutrino mass m_3. As one can see, the approximate equality of $|m_{ee}|$ and m_3, deduced for $z = 0$ in eq. (3.18), still holds for $z \neq 0$. The dashed (red) line is the result for $z = 0$. One finds that m_3 has a minimal value around $0.015\,\mathrm{eV}$, i.e. m_3 cannot vanish, and for $z = 0$ it also has a maximal one around $0.027\,\mathrm{eV}$. These two bounds can be found as well by using eq. (3.16).

The non-vanishing of $m_3 \approx |m_{\rm ee}|$ agrees with the findings in the literature that $|m_{\rm ee}|$ is required to be larger than 0.01 eV, if neutrinos follow an inverted hierarchy [24]. Figure 2 shows that the relation in eq. (3.16), which is fulfilled to a good accuracy for $z=0$, gives a lower bound for $z \neq 0$ in the $\tan\theta_{12}$-m_3 plane and no further constraints on the solar mixing angle can be derived. Note that we used the best-fit value of the atmospheric mass squared difference for the dashed (red) line in figure 2. Finally, we plot the Majorana phases ϕ_1 and ϕ_2 in figure 3 against the lightest neutrino mass m_3. As one can see, the phase ϕ_1 (blue/darker gray) varies between $\pi/8$ and $7\pi/8$, while ϕ_2 (green/lighter gray) either lies in the interval $[0, \pi/8]$ or $[7\pi/8, \pi]$ for small values of m_3, i.e. $m_3 \lesssim 0.06$ eV. The dashed (red) lines indicate again the value of ϕ_1 and ϕ_2 achieved in the limit $z=0$. As the difference $\phi_1 - \phi_2$ of the two Majorana phases is the only quantity which can be realistically determined by future experiments [22] through

$$|m_{\rm ee}| = |m_1 \cos^2\theta_{12} e^{2i\,(\phi_1-\phi_2)} + m_2 \sin^2\theta_{12}| \,, \tag{3.24}$$

we also plot $\phi_1 - \phi_2$ against m_3 in figure 4. This plot shows that the phase difference has to lie in the rather narrow ranges $[-3\pi/4, -\pi/2]$ or $[\pi/2, 3\pi/4]$ for small values of m_3. As one can see, the deviations from $|\phi_1 - \phi_2| = \pi/2$ ($z=0$ case) become larger for larger values of m_3.

3.3 Flavon superpotential

In the following we discuss the flavon superpotential and show that the VEV structure assumed in (eq. (3.2) and) eq. (3.3) naturally arises, as does the spontaneous CP violation. In constructing the superpotential we work along the lines of [12]. For this purpose, we generalize R-parity to a $\mathrm{U}(1)_R$ symmetry under which the "matter fields" transform with charge $+1$, the fields h_u and h_d and the flavons are uncharged and another type of fields, the driving fields, have charge $+2$. These fields transform trivially under the Standard Model gauge group, but non-trivially under the flavor symmetry. The set needed for constructing the potential consists of $\chi^0_e \sim (\underline{\mathbf{1}}_{\mathbf{1}}, \omega^4)$, $\sigma^0 \sim (\underline{\mathbf{1}}_{\mathbf{4}}, \omega)$ and $\chi^0_\nu \sim (\underline{\mathbf{1}}_{\mathbf{1}}, \omega)$ under (D_4, Z_5). Since all terms of the superpotential have to have $\mathrm{U}(1)_R$ charge $+2$, the driving fields cannot couple to the fermions and can only appear linearly in the flavon superpotential. The renormalizable $D_4 \times Z_5$ invariant superpotential for flavons and driving fields reads

$$\begin{aligned} w_f &= a\,\chi^0_e\,\chi^2_e + b\,\chi^0_e\,\varphi^2_e \\ &\quad + c\,\sigma^0\,(\psi^2_1 - \psi^2_2) + d\,\chi^0_\nu\,\psi_1\,\psi_2 + e\,\chi^0_\nu\,\varphi^2_\nu + f\,\chi^0_\nu\,\chi^2_\nu \,. \end{aligned} \tag{3.25}$$

Assuming that the flavons acquire their VEVs in the supersymmetric limit we can use the F-terms of the driving fields to determine the vacuum structure of the flavons. The equations

$$\frac{\partial w_f}{\partial \chi^0_e} = a\,\chi^2_e + b\,\varphi^2_e = 0\,, \tag{3.26a}$$

$$\frac{\partial w_f}{\partial \sigma^0} = c\,(\psi^2_1 - \psi^2_2) = 0\,, \tag{3.26b}$$

$$\frac{\partial w_f}{\partial \chi^0_\nu} = d\,\psi_1\,\psi_2 + e\,\varphi^2_\nu + f\,\chi^2_\nu = 0\,, \tag{3.26c}$$

result in

$$\langle \chi_e \rangle = \pm i \sqrt{\frac{b}{a}} \langle \varphi_e \rangle \;, \quad \langle \psi_1 \rangle = \pm \langle \psi_2 \rangle \;, \quad \langle \chi_\nu \rangle = \pm i \sqrt{\frac{d \langle \psi_1 \rangle \langle \psi_2 \rangle + e \langle \varphi_\nu \rangle^2}{f}} \tag{3.27}$$

which can be re-written as

$$w_e = \pm i \sqrt{\frac{b}{a}} u_e \;, \quad \langle \psi_1 \rangle = \pm v \;, \quad w = \pm i \sqrt{\frac{d \langle \psi_1 \rangle \langle \psi_2 \rangle + e\, u^2}{f}} \;. \tag{3.28}$$

Note that the VEVs $\langle \varphi_e \rangle = u_e$, $\langle \psi_2 \rangle = v$ and $\langle \varphi_\nu \rangle = u$ are unconstrained by the potential. Note further that the choice of sign in all cases is independent in eq. (3.27) and eq. (3.28). For the discussion of the preserved subgroup structure it is anyway only relevant whether $\langle \psi_1 \rangle = \langle \psi_2 \rangle$ or $\langle \psi_1 \rangle = -\langle \psi_2 \rangle$. For $\langle \psi_1 \rangle = \langle \psi_2 \rangle$ as used in eq. (3.3) we conserve a subgroup Z_2 of D_4 generated by B, whereas the relation $\langle \psi_1 \rangle = -\langle \psi_2 \rangle$ indicates that the Z_2 subgroup generated by BA^2 is left unbroken. This Z_2 group is also not a subgroup of the D_2 group conserved in the charged lepton sector. Thus, the subgroups of the charged lepton and the neutrino sector will be misaligned in both cases. In this paper we only consider the case of $\langle \psi_1 \rangle = \langle \psi_2 \rangle = v$. Eq. (3.28) shows then that the VEVs w_e and w necessarily have to be imaginary, so that CP is spontaneously violated, if the parameters $a, \dots, f$ and the VEVs u_e, v and u are chosen as positive.

We remark that due to the $\mathrm{U}(1)_R$ symmetry a μ-term $\mu h_u h_d$ is forbidden in our model and has to be generated by some other mechanism. This feature is shared by all models using a $\mathrm{U}(1)_R$ symmetry. In the derivation of eq. (3.26) terms of the form $\chi^0_\nu h_u h_d$ which couple a driving field to the MSSM Higgs fields can be safely neglected. They also cannot induce a μ-term, since only vanishing VEVs are allowed for the driving fields, if the parameters $a, \dots, f$ and the flavon VEVs are non-zero, as it is in our case. Finally, note that we find flat directions in this potential in the case of spontaneous CP violation under discussion here. These are however expected to be lifted by the inclusion of the NLO corrections, see section 4.2, as well as through soft supersymmetry breaking terms. Such nearly flat directions might be of interest for inflation [25].

4 Next-to-Leading Order corrections

In order to determine how our results are corrected at NLO, we take into account the effects of operators which are suppressed by one more power of the cutoff scale Λ compared to the LO. Such contributions to the fermion masses include two instead of only one flavon. In the flavon superpotential we add terms consisting of one driving field and three flavons. It turns out that there are actually no contributions to the fermion masses from two-flavon insertions due to the Z_5 symmetry. Hence, the only NLO corrections we need to consider are those of the flavon superpotential, which lead to a shift in the flavon VEVs parameterized as

$$\langle \chi_e \rangle = w_e + \delta w_e \;, \quad \langle \chi_\nu \rangle = w + \delta w \quad \text{and} \quad \langle \psi_1 \rangle = v + \delta v \;. \tag{4.1}$$

The VEVs $\langle \varphi_e \rangle = u_e$, $\langle \varphi_\nu \rangle = u$ and $\langle \psi_2 \rangle = v$ which are not determined at LO remain unconstrained also at NLO. The natural size of the VEV shifts is

$$\frac{\delta \mathrm{VEV}}{\mathrm{VEV}} \sim \lambda^2 \, . \tag{4.2}$$

As will be discussed in section 4.2, the shifts δw and δw_e are in general complex, whereas the shift δv in the VEV $\langle \psi_1 \rangle$ is real for this type of spontaneous CP violation.

4.1 Fermion masses

The VEV shifts induce corrections to the lepton mass matrices given in eq. (3.4) when the shifted VEVs are inserted into the LO terms, see eq. (3.1). In case of the charged lepton masses only the VEV of χ_e is shifted. Such a shift is however not relevant, since it can be absorbed into the Yukawa couplings y_1^e and y_2^e. [6] Especially, U_l is still given by eq. (3.5). The form of the neutrino mass matrix is changed through the shifts of the VEVs into

$$M_\nu = \frac{v_u^2}{\Lambda^2} \begin{pmatrix} y_1(w+\delta w) & y_2 v & y_2(v+\delta v) \\ y_2 v & y_3 u & y_4(w+\delta w) \\ y_2(v+\delta v) & y_4(w+\delta w) & y_3 u \end{pmatrix} . \tag{4.3}$$

Note that δw cannot be simply absorbed into w, since δw is complex, whereas w is imaginary. In the charged lepton mass basis the matrix in eq. (4.3) reads

$$M_\nu' = \frac{v_u^2}{\Lambda^2} \begin{pmatrix} y_1(w+\delta w) & y_2(v+e^{i\pi/4}\delta v/\sqrt{2}) & y_2(v+e^{-i\pi/4}\delta v/\sqrt{2}) \\ y_2(v+e^{i\pi/4}\delta v/\sqrt{2}) & y_4(w+\delta w) & y_3 u \\ y_2(v+e^{-i\pi/4}\delta v/\sqrt{2}) & y_3 u & y_4(w+\delta w) \end{pmatrix} . \tag{4.4}$$

To evaluate the shifts in the neutrino masses and to discuss the deviations of the mixing angles from their LO values, especially θ_{13} from zero and θ_{23} from maximal, we parameterize the Majorana neutrino mass matrix as

$$M_\nu' = \frac{v_u^2}{\Lambda} \frac{v}{\Lambda} \begin{pmatrix} i\, s\,(1+\alpha\,\epsilon) & t\,(1+e^{i\pi/4}\,\epsilon) & t\,(1+e^{-i\pi/4}\,\epsilon) \\ t\,(1+e^{i\pi/4}\,\epsilon) & i\,x\,(1+\alpha\,\epsilon) & z \\ t\,(1+e^{-i\pi/4}\,\epsilon) & z & i\,x\,(1+\alpha\,\epsilon) \end{pmatrix} \tag{4.5}$$

with s, t, x and z as given in eq. (3.12) and [7]

$$\alpha\,\epsilon = \frac{\delta w}{w} \; , \quad \alpha = \alpha_r + i\,\alpha_i \quad \text{and} \quad \epsilon = \frac{1}{\sqrt{2}} \frac{\delta v}{v} \approx \lambda^2 \approx 0.04 \, . \tag{4.6}$$

The neutrino masses and mixing parameters resulting from eq. (4.5) can then be calculated in an expansion in the small parameter ϵ. We observe that the mass shift of m_3^2 would vanish for δw being zero. Its explicit form is

$$(m_3^{\mathrm{NLO}})^2 = (m_3^{\mathrm{LO}})^2 + 2\left(\frac{v_u^2}{\Lambda}\right)^2 \left(\frac{v}{\Lambda}\right)^2 x(\alpha_r\, x + \alpha_i\, z)\,\epsilon \tag{4.7}$$

[6] These then become complex which however does not affect our results.
[7] We assume that ϵ is positive.

with $(m_3^{\rm LO})^2$ given in eq. (3.13). Similarly, the masses m_1^2 and m_2^2 undergo shifts proportional to ϵ. A simple expression can however only be found for the sum $m_1^2 + m_2^2$

$$(m_1^{\rm NLO})^2+(m_2^{\rm NLO})^2 = (m_1^{\rm LO})^2+(m_2^{\rm LO})^2+2\left(\frac{v_u^2}{\Lambda}\right)^2\left(\frac{v}{\Lambda}\right)^2(2\sqrt{2}\,t^2+\alpha_r\,(s^2+x^2)-\alpha_i\,x\,z)\,\epsilon \quad (4.8)$$

$(m_{1,2}^{\rm LO})^2$ can be found in eq. (3.13). The mixing angle θ_{13} no longer vanishes and we find

$$\sin\theta_{13} \approx \left|\frac{t\,x}{t^2+(s-x)x-z^2}\right|\epsilon\ . \quad (4.9)$$

For θ_{23} we get

$$\tan\theta_{23} \approx 1+\sqrt{2}\,\frac{x\,z}{t^2+(s-x)x-z^2}\,\epsilon\ . \quad (4.10)$$

The deviation from maximal atmospheric mixing can also be expressed through

$$|\cos 2\theta_{23}| \approx \sqrt{2}\left|\frac{x\,z}{t^2+(s-x)x-z^2}\right|\epsilon \approx \sqrt{2}\left|\frac{z}{t}\right|\sin\theta_{13}\ . \quad (4.11)$$

From both formulae one can deduce that in the case $z = 0$ the corrections to maximal atmospheric mixing are not of the order ϵ, but only arise at $\mathcal{O}(\epsilon^2)$. Contrary to this θ_{13} still receives corrections of order ϵ, if $z = 0$. The solar mixing angle θ_{12} which is not fixed to a precise value in this model also gets corrections of order ϵ. We note that the smallness of $|s-x|$ and z, required by the smallness of Δm_{21}^2, might lead to a disturbance of the expansion in the parameter ϵ.

A correlation between $\cos 2\theta_{23}$ and $\sin\theta_{13}$ depending only on physical quantities, Δm_{ij}^2, ..., and not on the parameters of the model, s, t, ..., can be obtained by an analytic consideration which is done analogously to the study performed in [7]. Clearly, the matrix in eq. (4.4) is no longer $\mu-\tau$ symmetric, however we find the following remnants of this symmetry

$$(M_\nu')_{e\mu} = (M_\nu')_{e\tau}^* \quad \text{and} \quad (M_\nu')_{\mu\mu} = (M_\nu')_{\tau\tau}\ . \quad (4.12)$$

Eq. (4.12) shows that $\mu-\tau$ symmetry is only broken by phases, but not by the absolute values of the matrix elements. This leads to

$$0 = |(M_\nu')_{e\mu}|^2+|(M_\nu')_{\mu\mu}|^2-|(M_\nu')_{e\tau}|^2-|(M_\nu')_{\tau\tau}|^2 \quad (4.13)$$

$$0 = (M_\nu' M_\nu'^\dagger)_{\mu\mu}-(M_\nu' M_\nu'^\dagger)_{\tau\tau} = \sum_{j=1}^{3} m_j^2(|(U_{\rm MNS})_{\mu j}|^2-|(U_{\rm MNS})_{\tau j}|^2)$$

$$\begin{aligned}0 = &\Big((\sin^2\theta_{12}-\sin^2\theta_{13}\cos^2\theta_{12})m_1^2+(\cos^2\theta_{12}-\sin^2\theta_{13}\sin^2\theta_{12})\,m_2^2-\cos^2\theta_{13}m_3^2\Big)\cos(2\theta_{23})\\ &-\Delta m_{21}^2\sin 2\theta_{12}\sin 2\theta_{23}\cos\delta\sin\theta_{13}\ . \end{aligned} \quad (4.14)$$

Since $\sin\theta_{13} \sim \mathcal{O}(\epsilon)$ and $\cos 2\theta_{23} \sim \mathcal{O}(\epsilon)$ is already known, we can linearize this equation and obtain (using best-fit values for the physical quantities and the fact that neutrinos have an inverted hierarchy in this model)

$$\cos 2\theta_{23} \approx -\frac{\Delta m_{21}^2\sin 2\theta_{12}}{\Delta m_{32}^2+\Delta m_{21}^2\sin^2\theta_{12}}\cos\delta\sin\theta_{13} \approx 0.03\cos\delta\sin\theta_{13}\ . \quad (4.15)$$

Eq. (4.15) can be used to estimate the largest possible deviation from maximal mixing. For $\sin\theta_{13}$ being at its $2\,\sigma$ limit of 0.2 and $|\cos\delta| = 1$, $|\cos 2\theta_{23}|$ still has to be less than 6×10^{-3} which is well within the $1\,\sigma$ error. Finally, we note that eq. (4.15) must be consistent with eq. (4.11) and thus we again find that z ought to be small.

4.2 Flavon superpotential

The corrections to the flavon superpotential stem from terms involving one driving field and three flavons. These terms are non-renormalizable and suppressed by the cutoff scale Λ. We find

$$\begin{aligned}\Delta w_f = {} & \frac{k_1}{\Lambda}\,\chi_e^0\,\chi_\nu^3 + \frac{k_2}{\Lambda}\,\chi_e^0\,\chi_\nu\,\varphi_\nu^2 + \frac{k_3}{\Lambda}\,\chi_e^0\,\chi_\nu\psi_1\psi_2 + \frac{k_4}{\Lambda}\,\chi_e^0\,\varphi_\nu\,(\psi_1^2+\psi_2^2) \\ & + \frac{k_5}{\Lambda}\,\sigma^0\,\varphi_e\,\chi_e^2 + \frac{k_6}{\Lambda}\,\sigma^0\,\varphi_e^3 + \frac{k_7}{\Lambda}\,\chi_\nu^0\,\chi_e^3 + \frac{k_8}{\Lambda}\,\chi_\nu^0\,\chi_e\varphi_e^2\,. \end{aligned} \tag{4.16}$$

Assuming that CP is only spontaneously violated forces all k_i to be real. We calculate the F-terms of $w_f + \Delta w_f$ for the driving fields using that the VEVs can be parameterized as

$$\langle\chi_e\rangle = w_e + \delta w_e\ , \quad \langle\chi_\nu\rangle = w + \delta w \quad \text{and} \quad \langle\psi_1\rangle = v + \delta v\ . \tag{4.17}$$

The VEVs $\langle\varphi_e\rangle = u_e$, $\langle\varphi_\nu\rangle = u$ and $\langle\psi_2\rangle = v$ are not determined at LO. We assume that only terms containing up to one VEV shift or the suppression factor $1/\Lambda$, but not both, are relevant. The F-terms then lead to

$$2\,a\,w_e\,\delta w_e + \frac{1}{\Lambda}(k_1\,w^3 + k_2\,u^2\,w + k_3\,v^2\,w + 2\,k_4\,u\,v^2) = 0\ , \tag{4.18a}$$

$$2\,c\,v\,\delta v + \frac{u_e}{\Lambda}(k_5\,w_e^2 + k_6\,u_e^2) = 0\ , \tag{4.18b}$$

$$d\,v\,\delta v + 2\,f\,w\,\delta w + \frac{w_e}{\Lambda}(k_7\,w_e^2 + k_8\,u_e^2) = 0\ . \tag{4.18c}$$

Here we have chosen the solutions with + in eq. (3.28). The explicit form of the shifts reads

$$\delta v = -\frac{1}{2\,c}\,\frac{u_e}{v\,\Lambda}\,(k_5\,w_e^2 + k_6\,u_e^2)\ , \tag{4.19}$$

$$\delta w = \frac{1}{4\,c\,f}\,\frac{1}{w\,\Lambda}\,(d\,(k_5 w_e^2 + k_6 u_e^2)\,u_e - 2\,c\,(k_7\,w_e^2 + k_8\,u_e^2)\,w_e)\ , \tag{4.20}$$

$$\delta w_e = -\frac{1}{2\,a}\,\frac{1}{w_e\,\Lambda}\,(k_1\,w^3 + k_2\,u^2\,w + k_3\,v^2\,w + 2\,k_4\,u\,v^2)\ . \tag{4.21}$$

As one can see, for our type of spontaneous CP violation δv is real, whereas δw_e and δw turn out to be complex in general. As can be read off from eq. (4.19) all shifts are generically of order

$$\frac{\delta\text{VEV}}{\text{VEV}} \sim \lambda^2 \quad \text{for} \quad \text{VEV} \sim \lambda^2\,\Lambda\ . \tag{4.22}$$

Finally, note that the free parameters $\langle\varphi_e\rangle = u_e$, $\langle\varphi_\nu\rangle = u$ and $\langle\psi_2\rangle = v$ are still undetermined.

5 Summary and outlook

We constructed a supersymmeterized version of the D_4 model by Grimus and Lavoura [1]. For this purpose, we replaced the Higgs doublets transforming under the flavor group D_4 by gauge singlets. We also enlarged the auxiliary symmetry which separates the different flavor breaking sectors from $Z_2^{(aux)}$ to Z_5. The simplest supersymmeterized D_4 model does not contain right-handed neutrinos, but neutrinos get masses through the operator $lh_u lh_u/\Lambda$. Apart from these slight changes the model is essentially the same as the one by GL, since we also generate maximal atmospheric mixing and vanishing θ_{13} through the fact that D_4 is broken to D_2 in the charged lepton and to Z_2 in the neutrino sector. The crucial issue of the vacuum alignment is elegantly solved through an appropriately constructed flavon potential. We performed a phenomenological analysis under the assumption of a certain type of spontaneous CP violation suggested by the minimization of the potential. As a result, the neutrinos have to have an inverted hierarchy. The quantity $|m_{\mathrm{ee}}|$, measured in $0\nu\beta\beta$ decay, is almost equal to the lightest neutrino mass m_3. Furthermore, we found that m_3 cannot vanish and has a lower bound around 0.015 eV. The Majorana phases ϕ_1 and ϕ_2 are restricted to a certain range at least for small m_3. In contrast to that the solar mixing angle θ_{12} can take all values allowed by experiments. We also analyzed the NLO terms in this model and showed that they only induce shifts in the VEVs of the flavons, but no additional terms in the Yukawa sector appear. The shifts yield deviations from the LO results, $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. Comparing these deviations we see that although both of them could in principle be of order $\epsilon \approx \lambda^2 \approx 0.04$, the smallness of the parameter z, necessary to arrive at mass squared differences and θ_{12} within the 2σ ranges, leads to the fact that θ_{23} is much closer to $\pi/4$ than θ_{13} to zero.

The supersymmeterization of the D_4 model has to be regarded as a step towards a grand unified model with a dihedral flavor symmetry for two reasons: (a) low scale supersymmetry elegantly cures the hierarchy problem and easily allows the gauge couplings to be unified at 10^{16} GeV and (b) the replacement of the Higgs doublets transforming under the flavor group by flavons is important for disentangling the breaking of flavor and gauge groups. However, another essential feature of a unified theory is that the lepton sector cannot be discussed without also considering quarks. So, one of the major challenges to tackle is the question how to implement the quark masses and their mixings in a model with a dihedral symmetry. In the recent past models have been presented which are able to predict the Cabibbo angle with the help of the flavor group D_7 and D_{14} [8–10]. Thus, it is interesting to search for a way to combine these models and to find a (probably larger) dihedral symmetry leading to the same results, which we get from a D_4 flavor group in the lepton sector and from a D_7 or D_{14} group in the quark sector.

Acknowledgments

We would like to thank W. Rodejohann and M. A. Schmidt for discussions. A.B. acknowledges support from the Studienstiftung des Deutschen Volkes. C.H. was supported by the "Sonderforschungsbereich" TR27.

A Connection to the group basis chosen in [1]

Note that replacing the left-handed fields l by $U_l^\star\, l$, with U_l given in eq. (3.5), is equivalent to changing the basis in which the generators A and B are given for the two-dimensional representation of D_4. Since also the second and third generation of the right-handed charged leptons form a doublet under D_4, we also have to transform them to show that this corresponds to a change of the generator basis of the D_4 doublet. By calculating the matrix U_{e^c} which diagonalizes $M_l^\dagger M_l$ one finds that U_{e^c} equals $U_l^\star\, P$ where P is a diagonal phase matrix. This matrix P induces an unphysical rephasing of the right-handed fields to keep $U_l^\dagger\, M_l\, U_{e^c}$ a diagonal matrix with positive entries. The change of basis (induced by the unitary matrix $U_l^\star$) leads to real generators A and B of the form

$$\mathrm{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{A.1}$$

This coincides with the basis chosen in [1], if A is identified with the product hg and B with the generator h. The identifications for the singlets are the following: $\underline{\mathbf{1}}_{++}$ corresponds to $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{1}}_{+-}$ to $\underline{\mathbf{1}}_{\mathbf{4}}$, $\underline{\mathbf{1}}_{-+}$ to $\underline{\mathbf{1}}_{\mathbf{3}}$ and $\underline{\mathbf{1}}_{--}$ to $\underline{\mathbf{1}}_{\mathbf{2}}$. In [1] the charged lepton mass matrix is diagonal without any further transformation and $\theta_{13} = 0$ and θ_{23} maximal can be directly read off from the neutrino mass matrix. This is the same in our case, if we go into the primed basis, see M_ν' in eq. (3.8).

B Importance of mismatch of subgroups

To elucidate the reason why the two subgroups preserved in the charged lepton and the neutrino sector have to be different, i.e. the Z_2 subgroup present in the neutrino sector should not be a subgroup of the D_2 group of the charged lepton sector, observe that $M_l\, M_l^\dagger$ as well as $M_\nu\, M_\nu^\dagger$ for M_l and M_ν given in eq. (3.4) can be written in the following form

$$M_i\, M_i^\dagger = \begin{pmatrix} A_i & B_i\, e^{i\,\beta_i} & B_i\, e^{i\,(\beta_i + \phi_i\,\mathrm{j})} \\ B_i\, e^{-i\,\beta_i} & C_i & D_i\, e^{i\phi_i\,\mathrm{j}} \\ B_i\, e^{-i\,(\beta_i+\phi_i\,\mathrm{j})} & D_i\, e^{-i\,\phi_i\,\mathrm{j}} & C_i \end{pmatrix} \qquad i = l, \nu\, . \tag{B.1}$$

This form is achieved for M_l (M_ν) as long as at least a Z_2 group, originating from $\mathrm{B\,A}^m$, is conserved in the charged lepton (neutrino) sector. A matrix of this type is diagonalized through

$$U_i = \begin{pmatrix} e^{i\,\beta_i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\,\phi_i\,\mathrm{j}} \end{pmatrix} U_{\mathrm{max}}\, U_{12}(\theta_i)\, U(\alpha_k^i) \tag{B.2}$$

where

$$U_{\mathrm{max}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} , \quad U_{12}(\theta_i) = \begin{pmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad U(\alpha_k^i) = \begin{pmatrix} e^{i\,\alpha_1^i} & 0 & 0 \\ 0 & e^{i\,\alpha_2^i} & 0 \\ 0 & 0 & e^{i\,\alpha_3^i} \end{pmatrix} . \tag{B.3}$$

In [9, 10] it has been shown that the quantities ϕ_i and j are related to the group theoretical indices of the flavor symmetry. j is the representation index of the doublet under which two of the three left-handed lepton generations transform. Thus, it is the same for charged leptons and neutrinos. ϕ_i can be expressed as

$$\phi_i = \frac{2\,\pi}{n}\, m_i \tag{B.4}$$

where n is the index of the group D_n and $m_{l\,(\nu)}$ the index of the preserved subgroup in the charged lepton (neutrino) sector that has a generator of the form $\mathrm{B}\,\mathrm{A}^{m_{l\,(\nu)}}$. $m_{l\,(\nu)}$ is an integer number between zero and $n-1$. The parameters A_i, ..., D_i and the phase β_i are real functions of the matrix entries of $M_i\, M_i^\dagger$, whose proper form is not needed here. The phases α^i_k are irrelevant for the diagonalization of $M_i\, M_i^\dagger$, but are necessary for the diagonalization of the neutrino mass matrix M_ν alone. The angle θ_i can be expressed through the parameters A_i, ..., D_i as follows

$$\tan 2\theta_i = \frac{2\sqrt{2}\, B_i}{C_i + D_i - A_i}\ . \tag{B.5}$$

The general form of the MNS matrix is then

$$U_{\mathrm{MNS}} = U_l^T\, U_\nu^\star = U(\alpha^l_k)\, U_{12}^T(\theta_l)\, U_{\mathrm{max}}^T \begin{pmatrix} e^{i\,(\beta_l-\beta_\nu)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\,(\phi_l-\phi_\nu)\,\mathrm{j}} \end{pmatrix} U_{\mathrm{max}}\, U_{12}(\theta_\nu)\, U(-\alpha^\nu_{\tilde{k}})\ . \tag{B.6}$$

This form already shows that it is essential to have a non-trivial phase $e^{-i\,(\phi_l-\phi_\nu)\,\mathrm{j}}$ in order to guarantee that the maximal mixing in the $2-3$ sector is not cancelled. For the third column of U_{MNS}, which determines the mixing angles θ_{13} and θ_{23}, we find

$$\begin{aligned} |(U_{\mathrm{MNS}})_{e3}| &= |\sin((\phi_l-\phi_\nu)\,\mathrm{j}/2)\, \sin\theta_l|\ , \quad |(U_{\mathrm{MNS}})_{\mu 3}| = |\sin((\phi_l-\phi_\nu)\,\mathrm{j}/2)\, \cos\theta_l|\ , \\ |(U_{\mathrm{MNS}})_{\tau 3}| &= |\cos((\phi_l-\phi_\nu)\,\mathrm{j}/2)|\ . \end{aligned} \tag{B.7}$$

Using that we preserve a Z_2 symmetry generated by B in the neutrino sector and a D_2 group generated by B A (according to our convention for the generators of the group D_2 introduced in section 2) in the charged lepton sector, gives for ϕ_ν and ϕ_l

$$\phi_\nu = 0 \quad \text{and} \quad \phi_l = \frac{\pi}{2}\ . \tag{B.8}$$

j is trivially one, since D_4 only contains one irreducible two-dimensional representation $\underline{\mathbf{2}}$. As the elements $(1,k)$ and $(k,1)$ with $k=2,3$ in M_l vanish, see eq. (3.4), the parameter B_l in eq. (B.1) is zero (and also $\beta_l = 0$) and thus $\theta_l = 0$ as well according to eq. (B.5). This results in

$$|(U_{\mathrm{MNS}})_{e3}| = 0\ , \quad |(U_{\mathrm{MNS}})_{\mu 3}| = |(U_{\mathrm{MNS}})_{\tau 3}| = \frac{1}{\sqrt{2}} \tag{B.9}$$

giving maximal atmospheric mixing and vanishing θ_{13}. A few things are interesting to notice: In principle four different cases might occur. These arise from whether the subgroups D_2 and Z_2 contain the same element BA^m or not and from whether the D_2 subgroup is unbroken in the charged lepton sector or only a Z_2 subgroup is preserved. The first issue determines whether m_l equals m_ν or not, i.e. whether $|\phi_l - \phi_\nu|$ is zero or not. The second one is responsible for (non-)zero θ_l. We can see from eq. (B.7) that for no mismatch of the subgroups θ_{13} as well as θ_{23} vanish, in contrast to what is observed in nature. So the mismatch of the two subgroups is necessary. If θ_l is zero, i.e. the subgroup present in the charged lepton sector is a D_2 group, $\theta_{13} = 0$ and θ_{23} maximal follow. If however only a smaller Z_2 group is present in the charged lepton sector, neither θ_{13} being zero nor θ_{23} being maximal holds. Then only the MNS matrix element $|(U_{\rm MNS})_{\tau 3}|$ is fixed by group theory.

Finally, the matrix U_l given in eq. (3.5) equals the matrix shown in eq. (B.2), if we additionally set the phases to $\alpha_1^l = 0$, $\alpha_2^l = \pi/4$ and $\alpha_3^l = 3\pi/4$.

One might ask the question what actually determines the size of the solar mixing angle θ_{12} in this context. For $\theta_l = 0$ we find from eq. (B.6) that

$$|(U_{\rm MNS})_{e1}| = |\cos\theta_\nu| \quad \text{and} \quad |(U_{\rm MNS})_{e2}| = |\sin\theta_\nu| \tag{B.10}$$

which shows that θ_{12} is given by θ_ν. Since this angle would vanish, if a D_2 group instead of a Z_2 group (with generator BA^m) was present in the neutrino sector, one might interpret the size of the solar mixing angle as hint to how strongly a D_2 group is broken in the neutrino sector.

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